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A new multi-ideal nano-topological model via neighborhoods for diagnosis and cure of dengue

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Abstract
The idea of nano-topology was originally proposed a decade ago by Thivagar. Since then, a lot of research has been done on the generalizations of the basic notions of nano-topology to overcome the limitations of an equivalence relation. The aim of this paper is to induce a novel frame of nano-topology using various covering-based neighborhoods via multiple ideals. The main properties of the proposed frame are acquired with the help of some illustrative instances, as well as its pros compared to the previous ones are investigated amply. A medical application is also discussed towards the end of this paper, where multi-ideal nano-topology is used to find the key symptoms of dengue disease. In addition, the most suitable medication is also suggested for the cure using the proposed theory.

Keywords Nano-topology · Approximation operator · Ideals · R_j-neighborhoods · Decision-making

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1 Introduction

First introduced in 1982 by Pawlak (1982), the rough set theory emphasizes the vagueness or uncertainty in any information system. Pawlak's study has proven to be quite effective

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in the mathematical modeling of quantitative data. The idea of a rough set was basically to generalize the general set theory by splitting it into two associated subsets, namely, upper approximation and lower approximation from a given relation on a set. Rough sets' theory is extensively used in decision-making, machine learning, data-mining, and processing of images as pointed out in many published manuscripts (Mareay 2016; Zhang et al. 2021; Zhong et al. 2003; Mareay 2024; El-Gayar 2022; El Sayed et al. 2021). In the past few decades, active research has been done on increasing the accuracy measure of approximating a rough set. In other words, the objective is to minimize the boundary regions by increasing the lower approximated version and reducing the upper approximated version. To increase the accuracy, the generalizations of a rough set were proposed and properties were studied, which remarkably helped to overcome the limitation of an equivalence relation; some of these generalizations are containment rough neighborhoods (Al-shami 2021b), maximal rough neighborhoods (Al-shami 2023), subset rough neighborhoods (Al-shami and Ciucci 2022), and recently cardinality rough neighborhoods (Al-shami and Hosny 2024). In this regard, it is worth noting that introducing the concept of idealization into rough set theory was intended to improve the properties of approximation operators and increase the value of the accuracy measure; that is, to reduce the region of uncertainty. Many researchers have exploited this approach to develop new models of rough set theory, as presented in published research such as Al-shami and Hosny (2022); Al-shami et al. (2023); Hosny et al. (2022).

Rough set models have been explored from a topological perspective since the similarities in the behaviors of topological and rough set concepts as discussed by Wiweger (1989). Later, a lot of topologists looked at the properties of rough sets by using the notions of topologies. To name a few, Lashin et al. (2005) dealt with neighborhoods as a subbase for topology and related the ideas of rough set theory and their counterparts in topology. Salama et al. (2022); Salama (2020, 2010) provided a topological approach to create lower and upper approximation operators under a family of dominance relations. Al-shami (Al-shami 2021a, 2022) exploited somewhere dense and somewhat dense subsets which are generalizations of open sets to extract the required information. Moreover, some authors employed supra topology (Al-shami and Alshammari 2023) and infra topology (Al-shami and Mhemdi 2023), and bitopology (Salama 2020), which are extensions of topology, to study generalized rough approximation spaces. Almarri and Azzam (2022) and El-Sharkasy (2021) discussed these spaces via the minimal structures.

Using the rough set theory as a base, nano-topology was introduced in 2013 by Thivagar and Richard (2013). As the name indicates, it is a "dwarf" topology because of its size. It was generated using approximation operators defined on a finite set with respect to an equivalence relation. Over the years, the theory has been extended to a binary relation, which can be utilized in a broader domain. Also, many researchers have worked on generating nano-topology using various mathematical tools like, neutrosophic sets (Thivagar et al. 2018), intuitionistic fuzzy sets (Al Shumrani et al. 2019; Gul et al. 2023; Malik et al. 2024, 2023), soft sets (Riaz et al. 2019), ideals (Kandil et al. 2013, 2020; Kaur and Gupta 2023, 2024), graphs (Arafa et al. 2020; Kaur and Gupta 2024), and neighborhoods (Abu-Gdairi et al. 2022; Al-shami and Hosny 2024; Kandil et al. 2021).

Nano-topology has numerous applications in various domains including science and technology. One remarkable application is in decision-making (Akram et al. 2019). Intriguingly, the rough set helps in the simplification and interpretation of any data set by extracting useful information, which is termed as "core". Likewise, the nano-topology serves the purpose which makes it a crucial tool for intelligent systems and information sciences. The nano-topological graph model has also helped in studying many biological processes and drawing imperative medical conclusions about the functioning of the human heart, fetal circulation, blood circulation, and urinary system (Arafa et al. 2020; Nawar and El Atik 2019). Also, this topology has helped in detecting the diseases like COVID-19 and lung disease (Azzam and Al-shami 2023; El-Bably and Al-shami 2021). In addition, nano-topology has also been applied to electric transmission lines (Nasef et al. 2020) for reduction and the simplification of electric circuits (El-Atik and Nasef 2020). In addition, the researchers have keenly been interested in studying the nearly open forms, continuity, separation axioms and other topological concepts of the different approaches of nano-topology (Kaur and Gupta 2024, 2023). This emerging subject has a great scope in the future and has great potential to serve in the biomedical field.

In the proposed paper, nano-topology is induced by multiple ideals (instead of single) via covering-based neighborhoods. Various results of this nano-topology are investigated with the aid of some counterexamples. Further, a comparison has been drawn to distinguish the current approach from the existing ones. Clearly, it is illustrated that the boundary of a set decreases and the accuracy measure increases when defined using the proposed approximation operators, so this notion is far better than the previous theories. It has been proven that Thivagar's definition (Thivagar and Richard 2013) and Kandil's definition (Kandil et al. 2021) are merely special cases of the proposed definition. Lastly, an algorithm is given to depict the significance of this multi-ideal nano-topology. Interestingly, this approach has been used to find the most significant symptoms of dengue disease. Moreover, the most suitable medicine has also been suggested for the cure of dengue disease from the given information table.

2 Preliminaries

The already established definitions and notations will be defined in this section which is required to deal with throughout this paper. Throughout this manuscript, U refers to the universal set and $\{l, r, i, u\} = \mathcal{Q}$, unless mentioned otherwise.

Definition 2.1 (Pawlak 1982) Let U be the universal set and $\tilde{R}$ be an indiscernibility relation. $\tilde{R}(s)$ is an equivalence class of s. The pair $(U, \tilde{R})$ is termed as an approximation space and $\tilde{Z} \subseteq U$, then the lower approximation, upper approximation and the boundary of $\tilde{Z}$ with respect to $\tilde{R}$ are respectively defined as:

1. $L_{\tilde{R}}(\tilde{Z}) = \bigcup_{s\in U} \left\{\tilde{R}(s) : \tilde{R}(s) \subseteq \tilde{Z}\right\}$.
2. $U_{\tilde{R}}(\tilde{Z}) = \bigcup_{s\in U} \left\{\tilde{R}(s) : \tilde{R}(s) \cap \tilde{Z} \neq \emptyset\right\}$.
3. $B_{\tilde{R}}(\tilde{Z}) = U_{\tilde{R}}(\tilde{Z}) - L_{\tilde{R}}(\tilde{Z})$.

Definition 2.2 (Thivagar and Richard 2013) Let $\tau_{\tilde{R}}(\tilde{Z}) = \left\{U, \emptyset, L_{\tilde{R}}(\tilde{Z}), U_{\tilde{R}}(\tilde{Z}), B_{\tilde{R}}(\tilde{Z})\right\}$, where $\tilde{Z} \subseteq U$. Then, $\tau_{\tilde{R}}(\tilde{Z})$ is known as nano-topology on U with respect to $\tilde{Z}$ and the pair $(U, \tau_{\tilde{R}}(\tilde{Z}))$ is called a nano-topological space. Elements of $(U, \tau_{\tilde{R}}(\tilde{Z}))$ are known as the nano-open sets, and their complements are called nano-closed sets.

Remark 2.3 (Thivagar and Richard 2013) If $K \subseteq U$, then the union of all nano-open subsets of K is called nano-interior of K, written as $\tau int(K)$ and the intersection of all nano-closed sets containing K is called nano-closure of K, written as $\tau cl(K)$.

Definition 2.4 (Kandil et al. 2021) Let U be the universe and $\tilde{R}$ be any relation on U. The after set and the fore set of $k \in U$ are respectively defined by $k\tilde{R} = \{y \in U;\ k\tilde{R}y\}$ and $\tilde{R}k = \{y \in U;\ y\tilde{R}k\}$.

Definition 2.5 (Abd El-Monsef et al. 2015) Let U be a non-empty finite set. Two types of covers of U induced from the relation $\tilde{R}$ are represented as follows:

1. $C_r = \{k\tilde{R} : k \in U\}$. If $\bigcup_{k\in U} k\tilde{R} = U$, then it is called r-cover of U.
2. $C_l = \{\tilde{R}k : k \in U\}$. If $\bigcup_{k\in U} \tilde{R}k = U$, then it is called l-cover of U.

Definition 2.6 (Abd El-Monsef et al. 2015) Let U be a non-empty finite set, let C_j be j-cover of U, where $j \in \{r, l\}$. The triplet $< U, \tilde{R}, C_j >$ is called $\tilde{R}_j$-covering approximation space (briefly, $\tilde{R}_j - C.A.S$).

Definition 2.7 (Abd El-Monsef et al. 2015) Let $< U, \tilde{R}, C_j >$ as the $\tilde{R}_j - C.A.S.$ $\forall\, k \in U$, the four neighborhoods $\tilde{R}_j(k)$, where $j \in \mathcal{Q}$ are defined as :

1. $\tilde{R}_r(k) = \bigcap\{\mathcal{F} \in C_r : k \in \mathcal{F}\}$.
2. $\tilde{R}_l(k) = \bigcap\{\mathcal{F} \in C_l : k \in \mathcal{F}\}$.
3. $\tilde{R}_i(k) = \tilde{R}_r(k) \cap \tilde{R}_l(k)$.
4. $\tilde{R}_u(k) = \tilde{R}_r(k) \cup \tilde{R}_l(k)$.

Lemma 2.8 (Kandil et al. 2021) Let $< U, \tilde{R}, C_j >$ as $\tilde{R}_j - C.A.S$ for all $j \in \mathcal{Q}$. Then,

1. $\tilde{R}_j(k) \neq \emptyset \;\forall\, k \in U$.
2. $k \in \tilde{R}_j(k) \;\forall\, k \in U$.
3. If $k \in \tilde{R}_j(y)$, then $\tilde{R}_j(k) \subseteq \tilde{R}_j(y)$.

Definition 2.9 (Kandil et al. 2021) Let $< U, \tilde{R}, C_j >$ as the $\tilde{R}_j - C.A.S$ for all $j \in \mathcal{Q}$ and $H \subseteq U$. Then, $\tilde{R}_j$- lower, $\tilde{R}_j$- upper approximation and $\tilde{R}_j$- boundary of a finite set H are defined as :

1. $\underline{\tilde{R}_j}(H) = \bigcup_{k\in U}\{\tilde{R}_j(k) : \tilde{R}_j(k) \subseteq H\}$.
2. $\overline{\tilde{R}_j}(H) = \bigcup_{k\in U}\{\tilde{R}_j(k) : \tilde{R}_j(k) \cap H \neq \emptyset\}$.
3. $B_{\tilde{R}_j}(H) = \overline{\tilde{R}_j}(H) - \underline{\tilde{R}_j}(H)$.

Definition 2.10 (Kandil et al. 2021) An ideal $\hat{\mathfrak{I}}$ on a set U is a non-empty collection of subsets of U for which the following holds true:

1. $H \in \hat{\mathfrak{I}}, K \subseteq H \implies K \in \hat{\mathfrak{I}}$.
2. $H \in \hat{\mathfrak{I}}, K \in \hat{\mathfrak{I}} \implies H \cup K \in \hat{\mathfrak{I}}$.

Definition 2.11 (Kandil et al. 2021) Let $< U, \tilde{R}, C_j >$ as the $\tilde{R}_j - C.A.S$ for all $j \in \mathcal{Q}$. Let $H \subseteq U$. If $\hat{\mathfrak{I}}$ is an ideal on U, then $\tilde{R}_j$- lower, $\tilde{R}_j$- upper approximation and $\tilde{R}_j$- boundary of a finite set H via an ideal $\hat{\mathfrak{I}}$ are defined as :

1. $\underline{\tilde{R}_j}^{\hat{\mathfrak{I}}}(H) = \{k \in H : \tilde{R}_j(k) \cap H^c \in \hat{\mathfrak{I}}\}$.
2. $\overline{\tilde{R}_j}^{\hat{\mathfrak{I}}}(H) = \big\{H \cup \{k \in U : \tilde{R}_j(x) \cap H \notin \hat{\mathfrak{I}}\}\big\}$.
3. $B^{\hat{\mathfrak{I}}}_{\tilde{R}_j}(H) = \overline{\tilde{R}_j}^{\hat{\mathfrak{I}}}(H) - \underline{\tilde{R}_j}^{\hat{\mathfrak{I}}}(H)$.

Definition 2.12 (Kandil et al. 2021) The collection $\tau^{\hat{\mathfrak{I}}}_{\tilde{R}_j}(H) = \{\emptyset,\ U,\ \overline{\tilde{R}_j}^{\hat{\mathfrak{I}}}(H),\ \underline{\tilde{R}_j}^{\hat{\mathfrak{I}}}(H), B^{\hat{\mathfrak{I}}}_{\tilde{R}_j}(H)\}$ forms a nano-topology via covering-based neighborhood induced by an ideal on U with respect to H for a binary relation $\tilde{R}$. The class $\tau^{\hat{\mathfrak{I}}}_{\tilde{R}_j}(H)$ is named a $\hat{\mathfrak{I}}$-nano-topology.

3 Covering based multi-ideal nano-topology

In this section, we define the covering-based multi-ideal nano-topology and derive some important results. Henceforth, R is a binary relation, unless mentioned otherwise.

Definition 3.1 Let $\hat{\mathfrak{I}}_1$, $\hat{\mathfrak{I}}_2$, $\hat{\mathfrak{I}}_3$,...,$\hat{\mathfrak{I}}_n$ be the 'n' ideals on U, then the collection induced by these 'n' ideals, denoted by $< n\hat{\mathfrak{I}} >$ is given by:

$$< n\hat{\mathfrak{I}} >= \{Q_1 \cup\ Q_2 \cup ... \cup Q_n : Q_1 \in \hat{\mathfrak{I}}_1, Q_2 \in \hat{\mathfrak{I}}_2, ..., Q_n \in \hat{\mathfrak{I}}_n\}.$$

Proposition 3.2 *If $\hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2, ..., \hat{\mathfrak{I}}_n$ are 'n' ideals on U, $H \subseteq U$ and $K \subseteq U$, then the set $< n\hat{\mathfrak{I}} >$ satisfies the following conditions:*

1. $< n\hat{\mathfrak{I}} >\neq \emptyset$.
2. $H \in< n\hat{\mathfrak{I}} >$ *and* $K \subseteq H \implies K \in< n\hat{\mathfrak{I}} >$.
3. $H, K \in< n\hat{\mathfrak{I}} > \implies H \cup K \in< n\hat{\mathfrak{I}} >$.

Proof 1. Since each $\hat{\mathfrak{I}}_i \neq \emptyset$, there exists a nonempty subset Q_i of U for each i, so $Q_1 \cup Q_2 \cup ... \cup Q_n$ is a nonempty subset of $\hat{\mathfrak{I}}_n$, which means that $< n\hat{\mathfrak{I}} >\neq \emptyset$.
2. Let $H \in< n\hat{\mathfrak{I}} >$. Then, by Definition 3.1 there exist some nonempty subsets Q_i of U that are contained in H. Without loss of generality, let $H = Q_1 \cup\ Q_2$. Suppose that $K \subseteq H$. Then it can divide K into subsets of Q_1 and Q_2; say, $K = M_1 \cup\ M_2$ such that $M_1 \subseteq Q_1$ and $M_2 \subseteq Q_2$. This implies that $M_1 \in \hat{\mathfrak{I}}_1$ and $M_2 \in \hat{\mathfrak{I}}_2$; thus, $K = M_1 \cup M_2 \cup \emptyset \cup ... \cup \emptyset \in< n\hat{\mathfrak{I}} >$, as required.
3. Following a similar argument given in item 2, the proof follows. □

According to the above proposition, $< n\hat{\mathfrak{I}} >$ is an ideal on U.

Definition 3.3 Let $< U, R, C_j >$ as the R_j - covering approximation space for all $j \in \mathcal{Q}$. Let $H \subseteq U$. If $\hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2, , \hat{\mathfrak{I}}_3, ..., \hat{\mathfrak{I}}_n$ are 'n' ideals on a non-empty set U, then the R_j- lower, R_j- upper approximation and R_j- boundary of a set H are mathematically represented as :

1. $\underline{R_j}^{<n\hat{\mathfrak{I}}>}(H) = \{k \in H : R_j(k) \cap H^c \in < n\hat{\mathfrak{I}} >\}$.
2. $\overline{R_j}^{<n\hat{\mathfrak{I}}>}(H) = H \cup \{k \in U : R_j(k) \cap H \notin < n\hat{\mathfrak{I}} >\}$.
3. $B^{<n\hat{\mathfrak{I}}>}{}_{R_j}(H) = \overline{R_j}^{<n\hat{\mathfrak{I}}>}(H) - \underline{R_j}^{<n\hat{\mathfrak{I}}>}(H)$.

Remark 3.4 If $\hat{\mathfrak{I}}_1 = \hat{\mathfrak{I}}_2 = \hat{\mathfrak{I}}_3 = ... = \hat{\mathfrak{I}}_n = \hat{\mathfrak{I}}$, then Definition 3.3 coincide with Definition 2.11 and if $\hat{\mathfrak{I}}_1 = \hat{\mathfrak{I}}_2 = \hat{\mathfrak{I}}_3 = ... = \hat{\mathfrak{I}}_n = \emptyset$, then the Definition 3.3 coincide with the Definition 2.9. If relation is equivalence and $\hat{\mathfrak{I}}_1 = \hat{\mathfrak{I}}_2 = \hat{\mathfrak{I}}_3 = ... = \hat{\mathfrak{I}}_n = \emptyset$, then the Definition 3.3 coincide with the Definition 2.1.

Definition 3.5 Let $< U, R, C_j >$ as R_j - covering approximation space. Then for $j \in \mathcal{Q}$, the accuracy measure of subset H of a finite universe U, induced by the different neighborhoods are respectively defined as :

$$\mu^{<n\hat{\mathfrak{I}}>}{}_{R_j}(H) = \frac{|\underline{R_j}^{<n\hat{\mathfrak{I}}>}(H)|}{|\overline{R_j}^{<n\hat{\mathfrak{I}}>}(H)|}$$

Definition 3.6 Let U be the universe, H be a subset of U, and R_j be the different kinds of neighborhoods, where $j \in \mathcal{Q}$. Then the class

$$\{U, \emptyset, \underline{R_j}^{<n\hat{\mathfrak{J}}>}(H),\ \overline{R_j}^{<n\hat{\mathfrak{J}}>}(H),\ B_{R_j}^{<n\hat{\mathfrak{J}}>}(H)\}$$

forms a topology that is termed as multi-ideal nano-topology, denoted by $\tau_{R_j}^{<n\hat{\mathfrak{J}}>}$ - NT.

The pair $(U, \tau_{R_j}^{<n\hat{\mathfrak{J}}>}(H))$ is defined as a multi-ideal nano-topological space, abbreviated as $\tau_{R_j}^{<n\hat{\mathfrak{J}}>} - NTS$.

Definition 3.7 The elements of $\tau_{R_j}^{<n\hat{\mathfrak{J}}>}(H)$ are called multi-ideal nano open ($\tau_{R_j}^{<n\hat{\mathfrak{J}}>}$-nano open) sets. The complements of $\tau_{R_j}^{<n\hat{\mathfrak{J}}>}$-nano open sets are called multi-ideal nano closed ($\tau_{R_j}^{<n\hat{\mathfrak{J}}>}$-nano closed) sets.

If $K \subseteq U$, then the union of all $\tau_{R_j}^{<n\hat{\mathfrak{J}}>}$-nano open subsets of K is called multi-ideal nano interior ($\tau_{R_j}^{<n\hat{\mathfrak{J}}>}$-nano interior) of K, written as $\tau_{R_j}^{<n\hat{\mathfrak{J}}>}int(K)$ and the intersection of all $\tau_{R_j}^{<n\hat{\mathfrak{J}}>}$-nano closed sets containing K is called multi-ideal nano closure ($\tau_{R_j}^{<n\hat{\mathfrak{J}}>}$-nano closure) of K, written as $\tau_{R_j}^{<n\hat{\mathfrak{J}}>}cl(K)$.

In what follows, we set up the main properties of the proposed approximation operators.

Proposition 3.8 *Let $(U, \tau_{R_j}{}^{<n\hat{\mathfrak{J}}>}(H))$ be an $\tau_{R_j}{}^{<n\hat{\mathfrak{J}}>} - NTS$ with respect to $H \subseteq U$ and let $E \cup F \subseteq U$. Then, the given results hold $\forall\ j \in \mathcal{Q}$, unless mentioned otherwise:*

1. $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(E) \subseteq E \subseteq \overline{R_j}^{<n\hat{\mathfrak{J}}>}(E)$.
2. $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(U) = \overline{R_j}^{<n\hat{\mathfrak{J}}>}(U) = U$.
3. $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(\emptyset)) = \overline{R_j}^{<n\hat{\mathfrak{J}}>}(\emptyset) = \emptyset$.
4. *If* $E \subseteq F$, *then* $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(E) \subseteq \underline{R_j}^{<n\hat{\mathfrak{J}}>}(F)$ *and* $\overline{R_j}^{<n\hat{\mathfrak{J}}>}(E) \subseteq \overline{R_j}^{<n\hat{\mathfrak{J}}>}(F)$.
5. $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(E) = (\overline{R_j}^{<n\hat{\mathfrak{J}}>}(E^c))^c$.
6. $\overline{R_j}^{<n\hat{\mathfrak{J}}>}(E) = (\underline{R_j}^{<n\hat{\mathfrak{J}}>}(E^c))^c$.
7. $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(\underline{R_j}^{<n\hat{\mathfrak{J}}>}(E)) = \underline{R_j}^{<n\hat{\mathfrak{J}}>}(E)$ *and* $\overline{R_j}^{<n\hat{\mathfrak{J}}>}(\overline{R_j}^{<n\hat{\mathfrak{J}}>}(E)) = \overline{R_j}^{<n\hat{\mathfrak{J}}>}(E))$.
8. $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(\underline{R_j}^{<n\hat{\mathfrak{J}}>}(E)) \subseteq \underline{R_j}^{<n\hat{\mathfrak{J}}>}(E)$ *and* $\overline{R_j}^{<n\hat{\mathfrak{J}}>} \subseteq \overline{R_j}^{<n\hat{\mathfrak{J}}>}(\overline{R_j}^{<n\hat{\mathfrak{J}}>}(E))$.
9. *If* $E \in <n\hat{\mathfrak{J}}>$, *then* $\overline{R_j}^{<n\hat{\mathfrak{J}}>}(E) = E$.
10. *If* $E^c \in <n\hat{\mathfrak{J}}>$, *then* $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(E) = E$.

Proof Proofs (1), (2), and (3) follow directly from Definition 3.3. (4) Let $E \subseteq F$. Let $v \in \underline{R_j}^{<n\hat{\mathfrak{J}}>}(E)$. Then, $v \in E$ and $R_j(v) \cap E^c \in <n\hat{\mathfrak{J}}>$. $\implies$ $v \in F$ and $R_j(v) \cap F^c \in <n\hat{\mathfrak{J}}>$ (as $E \subseteq F$). $\implies$ $v \in \underline{R_j}^{<n\hat{\mathfrak{J}}>}(F)$. Similarly, if $v \in \overline{R_j}^{<n\hat{\mathfrak{J}}>}(E)$, then $R_j(v) \cap E \notin <n\hat{\mathfrak{J}}>$. As $E \subseteq F$, $R_j(v) \cap F \notin <n\hat{\mathfrak{J}}>$ $\implies$ $v \in \overline{R_j}^{<n\hat{\mathfrak{J}}>}(F)$. Thus, $\underline{R_j}^{<n\hat{\mathfrak{J}}>}(E) \subseteq \underline{R_j}^{<n\hat{\mathfrak{J}}>}(F)\ \forall\ j \in \mathcal{Q}$.

(5) Let $v \in (\overline{R_j}^{<n\hat{\mathfrak{I}}>}((E^c)))^c \implies v \notin (\overline{R_j}^{<n\hat{\mathfrak{I}}>}(E^c)). \implies v \notin E^c$ and $v \notin \{w \in U : R_j(w) \cap E^c \notin < n\hat{\mathfrak{I}} >\}$.

Therefore, $v \in E$ and $R_j(w) \cap E^c \in < n\hat{\mathfrak{I}} >. \implies v \in \underline{R_j}^{<n\hat{\mathfrak{I}}>}(E)$.

For the converse part, $v \in \underline{R_j}^{<n\hat{\mathfrak{I}}>}(E), \implies v \in E$ and $R_j(v) \cap E^c \in < n\hat{\mathfrak{I}} >$. $\implies v \notin E^c$ and $v \notin \{w \in U : R_j(w) \cap E^c \notin < n\hat{\mathfrak{I}} >\}. \implies v \notin \overline{R_j}^{<n\hat{\mathfrak{I}}>}((E^c)$, that is, $v \in (\overline{R_j}^{<n\hat{\mathfrak{I}}>}(E^c))^c$. Therefore, $\underline{R_j}^{<n\hat{\mathfrak{I}}>}(E) = (\overline{R_j}^{<n\hat{\mathfrak{I}}>}(E^c))^c \ \forall \ j \in \mathcal{Q}$.

(6) Proof of this part is similar to (5).

(7) From parts (1) and (4), we have $\underline{R_j}^{<n\hat{\mathfrak{I}}>}(\underline{R_j}^{<n\hat{\mathfrak{I}}>}(E)) \subseteq \underline{R_j}^{<n\hat{\mathfrak{I}}>}(E)$. Now, we want to prove that $\underline{R_j}^{<n\hat{\mathfrak{I}}>}(E) \subseteq \underline{R_j}^{<n\hat{\mathfrak{I}}>}(\underline{R_j}^{<n\hat{\mathfrak{I}}>}(E))$. Let $v \notin \underline{R_j}^{<n\hat{\mathfrak{I}}>}(\underline{R_j}^{<n\hat{\mathfrak{I}}>}(E))$ for all $j \in \{l, r, i\}$. There exist two cases, firstly if $v \notin \underline{R_j}^{<n\hat{\mathfrak{I}}>}$, then $R_j(v) \cap (\underline{R_j}^{<n\hat{\mathfrak{I}}>}(E))^c \neq \emptyset$. Therefore, there exists $w \in R_j(v) \cap (\underline{R_j}^{<n\hat{\mathfrak{I}}>}(E))^c$. From Lemma 2.8, $R_j(w) \subseteq R_j(x)$ and $y \notin \underline{R_j}^{<n\hat{\mathfrak{I}}>}(E)$ for all $j \in \mathcal{Q}$. Hence, $w \notin E$ or $w \in E$, $R_j(w) \cap E^c \notin < n\hat{\mathfrak{I}} >$, that means $w \notin \underline{R_j}^{<n\hat{\mathfrak{I}}>}(E)$ or $R_j(v) \cap (E)^c \notin < n\hat{\mathfrak{I}} >$. Thus, both cases lead to $R_j(v) \cap (E)^c \notin < n\hat{\mathfrak{I}} >$, which is a contradiction with the assumption. This result need not be true for $j = u$, as depicted in the following counterexample: Let $U = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5\}$ be a universal set. $R = \{\{\Theta_1, \Theta_1\}, \{\Theta_1, \Theta_2\}, \{\Theta_1, \Theta_3\}, \{\Theta_3, \Theta_1\}, \{\Theta_2, \Theta_2\}, \{\Theta_3, \Theta_4\}, \{\Theta_5, \Theta_2\}, \{\Theta_4, \Theta_5\}\}$. It's a binary (non-equivalence) relation. Let $\hat{\mathfrak{I}}_1 = \{\emptyset, \{\Theta_1\}\}, \hat{\mathfrak{I}}_2 = \{\emptyset, \{\Theta_4\}\}$, then $< 2\hat{\mathfrak{I}} > \; = \; < \hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2 >= \{\emptyset, \{\Theta_1\}, \{\Theta_4\}, \{\Theta_1, \Theta_4\}\}$. If $E = \{\Theta_2, \Theta_3\}$, then $\underline{R_u}^{<n\hat{\mathfrak{I}}>}(E) = \{\Theta_3\}$ but $\underline{R_u}^{<n\hat{\mathfrak{I}}>}(\underline{R_u}^{<n\hat{\mathfrak{I}}>}(E)) = \emptyset$. Also, if $E = \{\Theta_1, \Theta_5\}$, $\overline{R_u}^{<n\hat{\mathfrak{I}}>}(E) = \{\Theta_1, \Theta_2, \Theta_5\}$ but $\overline{R_u}^{<n\hat{\mathfrak{I}}>}(\overline{R_u}^{<n\hat{\mathfrak{I}}>}(E)) = \{\Theta_1, \Theta_2, \Theta_3, \Theta_5\}$.

(8) By using (4), the proof of this part is obvious.

(9), (10) Proof follow directly from Definition 3.3. □

Proposition 3.9 *If the relation is symmetric, then* $\underline{R_r}^{<n\hat{\mathfrak{I}}>}(H) = \underline{R_l}^{<n\hat{\mathfrak{I}}>}(H) = \underline{R_i}^{<n\hat{\mathfrak{I}}>}(H) = \underline{R_u}^{<n\hat{\mathfrak{I}}>}(H)$ *and* $\overline{R_r}^{<n\hat{\mathfrak{I}}>}(H) = \overline{R_l}^{<n\hat{\mathfrak{I}}>}(H) = \overline{R_i}^{<n\hat{\mathfrak{I}}>}(H) = \overline{R_u}^{<n\hat{\mathfrak{I}}>}(H)$.

Proof It follows from the fact that $R_r(x) = R_l(x) = R_i(x) = R_u(x)$ for all $x \in U$ when R is symmetric. □

Example 3.10 Let $U = \{\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6, \Psi_7, \Psi_8\}$ and a subset $H = \{\Psi_1, \Psi_4, \Psi_6, \Psi_8\}$. Let $\hat{\mathfrak{I}}_1 = \{\emptyset, \{\Psi_2\}\}$, $\hat{\mathfrak{I}}_2 = \{\emptyset, \{\Psi_2\}, \{\Psi_6\}, \{\Psi_2, \Psi_6\}\}$, and $\hat{\mathfrak{I}}_3 = \{\emptyset, \{\Psi_5\}, \{\Psi_8\}, \{\Psi_5, \Psi_8\}\}$. Then, $< \hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2, \hat{\mathfrak{I}}_3 >= \{\emptyset, \{\Psi_2\}, \{\Psi_6\}, \{\Psi_5\}, \{\Psi_8\}, \{\Psi_2, \Psi_6\}, \{\Psi_6, \Psi_5\}, \{\Psi_5, \Psi_8\}, \{\Psi_2, \Psi_8\}, \{\Psi_2, \Psi_5\}, \{\Psi_6, \Psi_8\}, \{\Psi_2, \Psi_6, \Psi_5\}, \{\Psi_6, \Psi_5, \Psi_8\}, \{\Psi_5, \Psi_8, \Psi_2\}, \{\Psi_2, \Psi_6, \Psi_5, \Psi_8\}\}$. Let $R = \{\{\Psi_1, \Psi_1\}, \{\Psi_2, \Psi_2\}, \{\Psi_3, \Psi_3\}, \{\Psi_4, \Psi_4\}, \{\Psi_5, \Psi_5\}, \{\Psi_6, \Psi_6\}, \{\Psi_7, \Psi_7\}, \{\Psi_8, \Psi_8\}, \{\Psi_1, \Psi_2\}, \{\Psi_1, \Psi_5\}, \{\Psi_2, \Psi_6\}, \{\Psi_3, \Psi_8\}, \{\Psi_3, \Psi_1\}, \{\Psi_4, \Psi_8\}, \{\Psi_5, \Psi_2\}, \{\Psi_5, \Psi_7\}\}$. It is a non-equivalence relation (as it is non-symmetric and non-transitive). Then,

- $\tau_{R_r}^{<3\hat{\mathfrak{I}}>}(H) = \{\emptyset, U, \{\Psi_4, \Psi_6, \Psi_8\}, \{\Psi_1, \Psi_3, \Psi_4, \Psi_6, \Psi_8\}, \{\Psi_1, \Psi_3\}\}$.
- $\tau_{R_l}^{<3\hat{\mathfrak{I}}>}(H) = \{\emptyset, U, \{\Psi_1, \Psi_4, \Psi_6, \Psi_8\}, \{\Psi_1, \Psi_4, \Psi_6\}, \{\Psi_8\}\}$.
- $\tau_{R_i}^{<3\hat{\mathfrak{I}}>}(H) = \{\emptyset, U, \{\Psi_1, \Psi_4, \Psi_6, \Psi_8\}\}$.
- $\tau_{R_u}^{<3\hat{\mathfrak{I}}>}(H) = \{\emptyset, U, \{\Psi_6, \Psi_8\}, \{\Psi_1, \Psi_3, \Psi_4, \Psi_6, \Psi_8\}, \{\Psi_1, \Psi_3, \Psi_4\}\}$.

Note : So, in general, $\tau_{R_r}^{<n\hat{\mathfrak{I}}>}(H) \neq \tau_{R_l}^{<n\hat{\mathfrak{I}}>}(H) \neq \tau_{R_i}^{<n\hat{\mathfrak{I}}>}(H) \neq \tau_{R_u}^{<n\hat{\mathfrak{I}}>}(H)$, where H is a subset of U. (Because for non-symmetric relation, $R_r(x) \neq R_l(x) \neq R_i(x) \neq R_u(x)$ for all $x \in U$ and hence in general, $\underline{R_r}^{<n\hat{\mathfrak{I}}>}(H) \neq \underline{R_l}^{<n\hat{\mathfrak{I}}>}(H) \neq \underline{R_i}^{<n\hat{\mathfrak{I}}>}(H) \neq \underline{R_u}^{<n\hat{\mathfrak{I}}>}(H)$ and $\overline{R_r}^{<n\hat{\mathfrak{I}}>}(H) \neq \overline{R_l}^{<n\hat{\mathfrak{I}}>}(H) \neq \overline{R_i}^{<n\hat{\mathfrak{I}}>}(H) \neq \overline{R_u}^{<n\hat{\mathfrak{I}}>}(H)$.) In particular, $\tau_{R_r}^{<n\hat{\mathfrak{I}}>}cl(H) \neq \tau_{R_l}^{<n\hat{\mathfrak{I}}>}cl(H) \neq \tau_{R_i}^{<n\hat{\mathfrak{I}}>}cl(H) \neq \tau_{R_u}^{<n\hat{\mathfrak{I}}>}cl(H)$. Similarly, $\tau_{R_r}^{<n\hat{\mathfrak{I}}>}int(H) \neq \tau_{R_l}^{<n\hat{\mathfrak{I}}>}int(H) \neq \tau_{R_i}^{<n\hat{\mathfrak{I}}>}int(H) \neq \tau_{R_u}^{<n\hat{\mathfrak{I}}>}int(H)$, where H is a subset of U.

Theorem 3.11 *Let $(U, \tau_{R_j}^{<n\hat{\mathfrak{I}}>}(X))$ be an $\tau_{R_j}^{<n\hat{\mathfrak{I}}>} - NTS$ with respect to X where $X \subseteq U$. Then, the following results are true for every $j \in \mathcal{Q}$ and any subset K of U:*

1. $U - \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(K) = \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(U - K)$.
2. $U - \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(K) = \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(U - K)$.

Proof Obvious by the definition of $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}$-nano closure and $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}$-nano interior. □

The proof of the next two propositions is obvious by the definition of $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}$-nano closure and $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}$-nano interior.

Proposition 3.12 *Let $(U, \tau_{R_j}^{<n\hat{\mathfrak{I}}>}(H))$ be an $\tau_{R_j}^{<n\hat{\mathfrak{I}}>} - NTS$ with respect to H where $H \subseteq U$. and let $E \cup F \subseteq U$. Then the results are true for every $j \in \mathcal{Q}$ as following:*

1. $E \subseteq \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E)$.
2. *E is multi-ideal nano closed if* $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E) = E$.
3. $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(\emptyset) = \emptyset$ *and* $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(U) = U$.
4. $E \subseteq F \implies \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E) \subseteq \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(F)$.
5. $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E \cup F) = \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E) \cup \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(F)$.
6. $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E \cap F) \subseteq \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E) \cap \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(F)$.
7. $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(\tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E)) = \tau_{R_j}^{<n\hat{\mathfrak{I}}>}cl(E)$.

Proposition 3.13 *Let $(U, \tau_{R_j}^{<n\hat{\mathfrak{I}}>}(X))$ be an $\tau_{R_j}^{<n\hat{\mathfrak{I}}>} - NTS$ with respect to X where $X \subseteq U$. and let $E \cup F \subseteq U$. Then the results are true for every $j \in \mathcal{Q}$ as following:*

1. *E is multi-ideal nano open iff* $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(E) = E$.
2. $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(\emptyset) = \emptyset$ *and* $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(U) = U$.
3. $E \subseteq F \implies \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(E) \subseteq \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(F)$.
4. $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(E) \cup \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(F) \subseteq \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(E \cup F)$.
5. $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(E \cap F) = \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(E) \cap \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(F)$.
6. $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(\tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(E)) = \tau_{R_j}^{<n\hat{\mathfrak{I}}>}int(E)$.

Proposition 3.14 *Let $(U, \tau_{R_j}^{<n\hat{\mathfrak{I}}>}(X))$ be an $\tau_{R_j}^{<n\hat{\mathfrak{I}}>} - NTS$ with respect to X where $E \cup F \subseteq U$. Then the statements hold true $\forall\ j \in \mathcal{Q}$ as follows:*

1. $\underline{R_j}^{<n\hat{\Im}>}(E \cap F) = \underline{R_j}^{<n\hat{\Im}>}(E) \cap \underline{R_j}^{<n\hat{\Im}>}(F)$.
2. $\underline{R_j}^{<n\hat{\Im}>}(E) \cup \underline{R_j}^{<n\hat{\Im}>}(F) \subseteq \underline{R_j}^{<n\hat{\Im}>}(E \cup F)$.
3. $\overline{R_j}^{<n\hat{\Im}>}(E \cap F) \subseteq \overline{R_j}^{<n\hat{\Im}>}(E) \cap \overline{R_j}^{<n\hat{\Im}>}(F)$.
4. $\overline{R_j}^{<n\hat{\Im}>}(E \cup F) = \overline{R_j}^{<n\hat{\Im}>}(E) \cup \overline{R_j}^{<n\hat{\Im}>}(F)$.

Proof (1) Let $v \in \underline{R_j}^{<n\hat{\Im}>}(E) \cap \underline{R_j}^{<n\hat{\Im}>}(F)$. Then, $v \in E,\ R_j(v) \cap E^c \in <n\hat{\Im}>$ and $v \in F$, $R_j(v) \cap F^c \in <n\hat{\Im}>$. $\implies v \in E \cap F$ and $(R_j(v) \cap E^c) \cup (R_j(v) \cap F^c) \in <n\hat{\Im}>$. $\implies v \in E \cap F$ and $R_j(v) \cap (E \cap F)^c \in <n\hat{\Im}>$. $\implies v \in \underline{R_j}^{<n\hat{\Im}>}(E \cap F)$. Thus, $\underline{R_j}^{<n\hat{\Im}>}(E) \cap \underline{R_j}^{<n\hat{\Im}>}(F) \subseteq \underline{R_j}^{<n\hat{\Im}>}(E \cap F)$. For the converse part, $E \cap F \subseteq E$ and $E \cap F \subseteq F$. $\implies \underline{R_j}^{<n\hat{\Im}>}(E \cap F) \subseteq \underline{R_j}^{<n\hat{\Im}>}(E)$ and $\underline{R_j}^{<n\hat{\Im}>}(E \cap F) \subseteq \underline{R_j}^{<n\hat{\Im}>}(F)$. $\implies \underline{R_j}^{<n\hat{\Im}>}(E \cap F) \subseteq \underline{R_j}^{<n\hat{\Im}>}(E) \cap \underline{R_j}^{<n\hat{\Im}>}(F)$. So, $\underline{R_j}^{<n\hat{\Im}>}(E \cap F) = \underline{R_j}^{<n\hat{\Im}>}(E) \cap \underline{R_j}^{<n\hat{\Im}>}(F)$.
(2) Since, $E \subseteq E \cup F$ and $F \subseteq E \cup F$. So, $\underline{R_j}^{<n\hat{\Im}>}(E) \subseteq \underline{R_j}^{<n\hat{\Im}>}(E \cup F)$ and $\underline{R_j}^{<n\hat{\Im}>}(F) \subseteq \underline{R_j}^{<n\hat{\Im}>}(E \cup F)$. Hence, $\underline{R_j}^{<n\hat{\Im}>}(E) \cup \underline{R_j}^{<n\hat{\Im}>}(F) \subseteq \underline{R_j}^{<n\hat{\Im}>}(E \cup F)$.
(3) As $E \cap F \subseteq E$ and $E \cap F \subseteq F$. $\implies \overline{R_j}^{<n\hat{\Im}>}(E \cap F) \subseteq \overline{R_j}^{<n\hat{\Im}>}(E)$ and $\overline{R_j}^{<n\hat{\Im}>}(E \cap F) \subseteq \overline{R_j}^{<n\hat{\Im}>}(F)$. Therefore, $\overline{R_j}^{<n\hat{\Im}>}(E \cap F) \subseteq \overline{R_j}^{<n\hat{\Im}>}(E) \cap \overline{R_j}^{<n\hat{\Im}>}(F)$.
(4) As $E \subseteq E \cup F$ and $F \subseteq E \cup F$. $\implies \overline{R_j}^{<n\hat{\Im}>}(E) \subseteq \overline{R_j}^{<n\hat{\Im}>}(E \cup F)$ and $\overline{R_j}^{<n\hat{\Im}>}(F) \subseteq \overline{R_j}^{<n\hat{\Im}>}(E \cup F)$. $\implies \overline{R_j}^{<n\hat{\Im}>}(E) \cup \overline{R_j}^{<n\hat{\Im}>}(F) \subseteq \overline{R_j}^{<n\hat{\Im}>}(E \cup F)$. For the proof of converse part, let $v \notin \overline{R_j}^{<n\hat{\Im}>}(E) \cup \overline{R_j}^{<n\hat{\Im}>}(F)$. $\implies v \notin \overline{R_j}^{<n\hat{\Im}>}(E)$ and $v \notin \overline{R_j}^{<n\hat{\Im}>}(F)$. $\implies v \notin E$, $R_j(v) \cap E \in <n\hat{\Im}>$ and $v \notin F$, $R_j(v) \cap F \in <n\hat{\Im}>$. $\implies v \notin E \cup F$ and $R_j(v) \cap (E \cup F) \in <n\hat{\Im}>$. $\implies v \notin \overline{R_j}^{<n\hat{\Im}>}(E \cup F)$. $\implies \overline{R_j}^{<n\hat{\Im}>}(E \cup F) \subseteq \overline{R_j}^{<n\hat{\Im}>}(E) \cup \overline{R_j}^{<n\hat{\Im}>}(F)$. Therefore, $\overline{R_j}^{<n\hat{\Im}>}(E \cup F) = \overline{R_j}^{<n\hat{\Im}>}(E) \cup \overline{R_j}^{<n\hat{\Im}>}(F)$. □

Proposition 3.15 *Let $(U, \tau_{R_j}^{<n\hat{\Im}>}(X))$ be an $\tau_{R_j}^{<n\hat{\Im}>} - NTS$ with respect to X where $X \subseteq U$. Let $E \subseteq U$. Then the following statements hold true:*

1. $\underline{R_u}^{<n\hat{\Im}>}(E) \subseteq \underline{R_r}^{<n\hat{\Im}>}(E) \subseteq \underline{R_i}^{<n\hat{\Im}>}(E)$.
2. $\underline{R_u}^{<n\hat{\Im}>}(E) \subseteq \underline{R_l}^{<n\hat{\Im}>}(E) \subseteq \underline{R_i}^{<n\hat{\Im}>}(E)$.
3. $\overline{R_i}^{<n\hat{\Im}>}(E) \subseteq \overline{R_r}^{<n\hat{\Im}>}(E) \subseteq \overline{R_u}^{<n\hat{\Im}>}(E)$.
4. $\overline{R_i}^{<n\hat{\Im}>}(E) \subseteq \overline{R_l}^{<n\hat{\Im}>}(E) \subseteq \overline{R_u}^{<n\hat{\Im}>}(E)$.

Proof (1) Let $v \in \underline{R_u}^{<n\hat{\Im}>}(E)$, then $R_u(v) \cap E^c \in <n\hat{\Im}>$. $\implies (R_r(v) \cup R_l(v)) \cap E^c \in <n\hat{\Im}>$. $\implies R_r(v) \cap E^c \in <n\hat{\Im}>$. $\implies R_r(v) \cap E^c \in <n\hat{\Im}>$. $\implies v \in \underline{R_r}^{<n\hat{\Im}>}(E)$. Similarly, let $v \in \underline{R_r}^{<n\hat{\Im}>}(E) \implies R_r(v) \cap E^c \in <n\hat{\Im}>$. $\implies R_i(v) \cap E^c \in <n\hat{\Im}>$. (As $R_i(v) \subseteq R_r(v)$.)
(2) Proof of this part is similar to (1).

(3) For the proof, let $v \notin \overline{R_u}^{<n\hat{\Im}>}(E)$. $\implies$ $R_u(v) \cap E \in <n\hat{\Im}>$. $\implies$ $R_r(v) \cap E \in <n\hat{\Im}>$. $\implies$ $v \notin \overline{R_r}^{<n\hat{\Im}>}(E)$. So, $\overline{R_r}^{<n\hat{\Im}>}(E) \subseteq \overline{R_u}^{<n\hat{\Im}>}(E)$. Similarly, $\overline{R_i}^{<n\hat{\Im}>}(E) \subseteq \overline{R_r}^{<n\hat{\Im}>}(E)$. (4) Proof of this part is similar to (3). $\square$

4 A comparison of the introduced ideology with the already established ones

The term accuracy has a vital role in the decision-making process, wherein the theories of rough set or nano-topology are used in any information table or real-life situations. Remarkably, the suggested approach, which is based on multiple ideals to generate the nano-topology is way better than the previous ones because the generation of nano-topology through one ideal is just a particular case of this method, where all the ideals are the same. Also, the standard definition is a particular case of this definition if the domain of R is restricted to an equivalence relation and ideals contain necessarily an empty set only. In our technique, the lower approximation increases and the boundary decreases; therefore, leading to higher accuracy in comprehending the interrelation between conditional attributes and decision-related factors, that's why it has a great scope for furthermore research and physical importance. Also, as the proposed approach extends the standard definitions restricted to equivalence relation to a broader term; that is, binary relation, hence it is of great significance for future scope in the real world.

Some results are given below to show the priority of the proposed theory to the previous ones. If $\hat{\Im}_i, \hat{\Im}_2, ...\hat{\Im}_n$ are the 'n' ideals which generate ideal $<n\hat{\Im}>$, then the following are true $\forall\ j \in \mathcal{Q}$ and $\forall$ i=1,2,...,n:

1. $\overline{R_j}^{<n\hat{\Im}>}(Z) \subseteq \overline{R_j}^{\hat{\Im}_i}(Z) \subseteq \overline{R_j}(Z)$.
2. $\underline{R_j}(Z) \subseteq \underline{R_j}^{\hat{\Im}_i}(Z) \subseteq \underline{R_j}^{<n\hat{\Im}>}(Z)$.
3. $B^{<n\hat{\Im}>}{}_{R_j}(Z) \subseteq B^{\hat{\Im}_i}{}_{R_j}(Z) \subseteq B_R(Z)$.
4. $\mu(Z) \leq \mu^{\hat{\Im}_i}{}_{R_j}(Z) \leq \mu^{<n\hat{\Im}>}{}_{R_j}(Z)$.
   Note that μ refers to the accuracy, that is the ratio of the lower approximation to the upper one.
5. Multi-ideal nano-topology $\tau_{R_j}^{<n\hat{\Im}>}(H)$, single-ideal topology $\tau_{R_j}^{\hat{\Im}_i}(H)$, and standard term of nano-topology $\tau_R(H)$ are independent of each other.
6. Multi-ideal approximations are the same as the single ideal ones when $\hat{\Im}_1 = \hat{\Im}_2 = ...\hat{\Im}_n = \hat{\Im}$, so multi-ideal nano-topology is a generalization of the nano-topology, generated by a single ideal, $\tau_{R_j}^{\hat{\Im}}(H)$, as defined in Definition 2.12.
7. If the relation is equivalence and $\hat{\Im}_1 = \hat{\Im}_2 = ...\hat{\Im}_n = \emptyset$, then the multi-ideal nano-topology is same as the basic definition of nano-topology, that is $\tau_R(H)$, as defined in Definition 2.2.

In what follows, we mention the main advantages of the proposed research to highlight its significance and future scope.

This theory can also be applied to assess the credibility of expert reports, which are mathematically represented as ideals. As previously mentioned, this method combines $'n'$ ideals to create a multi-ideal nano-topology, which is then used to interpret data. The approximation of a set depends on the choice of ideals, as given in Definition 3.6. Given an information table containing a universal set U, parameters (conditional attributes) defining R, $'n'$ opinions (ideals), and a final decision (X). If $\hat{\mathfrak{I}}_1$,$\hat{\mathfrak{I}}_2$,...$\hat{\mathfrak{I}}_k$....$\hat{\mathfrak{I}}_n$ are the $'n'$ ideals, then $< n\hat{\mathfrak{I}} >$ is the ideal generated by all of them. We intend to compare the proficiency of the different experts. To demonstrate this, the following remarks are stated, followed by an example.

a. If R is an indiscernibility relation with respect to a single attribute/parameter such that $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}(H) = \tau_{R_j}^{\hat{\mathfrak{I}}_k}(H)$ for every j, then $\hat{\mathfrak{I}}_k$ is not an insignificant or irreliable report for assessing that specific parameter in comparison to $< n\hat{\mathfrak{I}} >$.
b. If $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}(H) = \tau_{R_j}^{\hat{\mathfrak{I}}_k}(H)$ for the relation R with respect to multiple attributes (taken one at a time), and $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}(H) = \tau_{R_j}^{\hat{\mathfrak{I}}_k}(H)$ for the relation R all attributes (taken simultaneously), then $\hat{\mathfrak{I}}_k$ is a reliable report for the overall information system.
c. If R is the relation with respect to all conditional attributes/ parameters and $\tau_{R_j}^{<n\hat{\mathfrak{I}}>}(H) \neq \tau_{R_j}^{\hat{\mathfrak{I}}_k}(H)$, then the $\hat{\mathfrak{I}}_k$ is not a reliable opinion, this report should be discarded and shouldn't be trusted independently for forming or analyzing a decision. This ideal (expert opinion) cannot be trusted to investigate the interrelation of conditional and decision attributes in information systems.
d. If R is the indiscernibility relation with respect to multiple attributes (taken one at a time as well as taken simultaneously) and the respective multi-ideal nano topologies satisfy $\tau_{R_j}^{<n\hat{\mathfrak{I}}>-\hat{\mathfrak{I}}_k}(H) = \tau_{R_j}^{<n\hat{\mathfrak{I}}>}$, then the $\hat{\mathfrak{I}}_k$ is totally insignificant opinion, this report should be discarded and not be considered sufficient for the data interpretation. Here, $< n\hat{\mathfrak{I}} > -\hat{\mathfrak{I}}_k$ is the ideal generated by $n-1$ ideals.

$$< n\hat{\mathfrak{I}} > -\hat{\mathfrak{I}}_k =< \hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2, ...\hat{\mathfrak{I}}_{k-1}, \hat{\mathfrak{I}}_{k+1}....\hat{\mathfrak{I}}_{\mathrm{n}} > .$$

e. If R is the relation with respect to multiple attributes (taken one at a time as well as taken simultaneously) and the respective multi-ideal nano topologies satisfy $\tau_{R_j}^{<n\hat{\mathfrak{I}}>-\hat{\mathfrak{I}}_k}(H) \neq \tau_{R_j}^{<n\hat{\mathfrak{I}}>}$, then the $\hat{\mathfrak{I}}_k$ is a significant opinion, this report shouldn't be neglected right away as it will influence the data analysis. Throughout, $\hat{\mathfrak{I}}_k \subseteq < n\hat{\mathfrak{I}} >$ for every $k = 1, 2, ...n$.

These pros can be observed from the following example (Example 4.1).

Example 4.1 Consider an example of six candidates competing for a job in Table 1. Let us suppose that three experts gave their opinions on the selection of candidates based on their performance in a preliminary test in an organization, which comprised general knowledge, academic scores, and interviewing skills. Finally, three candidates were selected for the job. So, based on the information, we compare the experts' opinions.

Let $U = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$, $H = \{\varsigma_1, \varsigma_3, \varsigma_6\}$, and $H^c = \{\varsigma_2, \varsigma_4, \varsigma_5\}$. The domain of the attribute *general knowledge* $=$ {good, medium, poor} (Note that good assessment is given to score 81–90, medium assessment is given to score 71–80, and poor evaluation is given to score 61–70. The domain of the attribute *academics* $=$ {excellent, average, weak} (Note that excellent assessment is given to score 76–85, medium assessment is given to score 66–75, and poor evaluation is given to score 56–65. The domain of the attribute *interview skills* $=$ {outstanding, mediocre, terrible}.

The domain of the decision attribute $= \{\checkmark, ✗\}$.

The indiscernibility relation (with respect to single attribute *"general knowledge"*) $R_1 = \{\{\varsigma_1, \varsigma_5\}, \{\varsigma_2\}, \{\varsigma_3, \varsigma_4, \varsigma_6\}\}$. The indiscernibility relation (with respect to the attribute *"academics"*) $R_2 = \{\{\varsigma_1, \varsigma_4, \varsigma_6\}, \{\varsigma_2, \varsigma_5\}, \{\varsigma_3\}\}$. The indiscernibility relation (with respect

Table 1 Illustration of candidates with respect to different parameters and decision

Candidates	General knowledge	Academics	Interview skills	Selection
ς_1	69 (poor)	85 (excellent)	Outstanding	✓
ς_2	80 (medium)	56 (weak)	Mediocre	✗
ς_3	82 (good)	67 (average)	Outstanding	✓
ς_4	82 (good)	76 (excellent)	Outstanding	✗
ς_5	65 (poor)	63 (weak)	Terrible	✗
ς_6	90 (good)	85 (excellent)	Outstanding	✓

to the attribute *"interview skills"*) $R_3 = \{\{\varsigma_1, \varsigma_3, \varsigma_4, \varsigma_6\}, \{\varsigma_5\}, \{\varsigma_2\}\}$. The relation (*with respect to all the attributes taken simultaneously*) $R_4 = \{\{\varsigma_4, \varsigma_6\}, \{\varsigma_1\}, \{\varsigma_2\}, \{\varsigma_3\}, \{\varsigma_5\}\}$.

Let three expert opinions be represented as $\hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2$, and $\hat{\mathfrak{I}}_3$. So, $\hat{\mathfrak{I}}_1 = \{\emptyset, \{\varsigma_3\}, \{\varsigma_4\}, \{\varsigma_3, \varsigma_4\}\}$, $\hat{\mathfrak{I}}_2 = \{\emptyset, \{\varsigma_1\}, \{\varsigma_6\}, \{\varsigma_1, \varsigma_6\}\}$, $\hat{\mathfrak{I}}_3 = \{\emptyset, \{\varsigma_3\}\}$. Then, ideal generated by $\hat{\mathfrak{I}}_1$, $\hat{\mathfrak{I}}_2$, and $\hat{\mathfrak{I}}_3$ is $< \hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2, \hat{\mathfrak{I}}_3 > = < 3\hat{\mathfrak{I}} > = \{\emptyset, \{\varsigma_1\}, \{\varsigma_3\}, \{\varsigma_4\}, \{\varsigma_6\}, \{\varsigma_1, \varsigma_3\}, \{\varsigma_1, \varsigma_4\}, \{\varsigma_1, \varsigma_6\}, \{\varsigma_3, \varsigma_4\}, \{\varsigma_3, \varsigma_6\}, \{\varsigma_4, \varsigma_6\}, \{\varsigma_1, \varsigma_3, \varsigma_4\}, \{\varsigma_1, \varsigma_3, \varsigma_6\}, \{\varsigma_3, \varsigma_4, \varsigma_6\}, \{\varsigma_1, \varsigma_4, \varsigma_6\}, \{\varsigma_1, \varsigma_3, \varsigma_4, \varsigma_6\}\}$.

Observation: For the relation R_4, where we consider indiscernibility with respect to all attributes taken together, then the respective ideal nano topologies

$$\tau_{R_4}^{<3\hat{\mathfrak{I}}>}(H) = \tau_{R_4}^{\hat{\mathfrak{I}}_1}(H) = \tau_{R_4}^{\hat{\mathfrak{I}}_2}(H) = \tau_{R_4}^{\hat{\mathfrak{I}}_3}(H).$$

On the other hand, when we consider R_1, R_2, R_3 (considering parameters, one at a time),

$$\tau_{R_q}^{<3\hat{\mathfrak{I}}>}(H) \neq \tau_{R_q}^{\hat{\mathfrak{I}}_1}(H) \neq \tau_{R_q}^{\hat{\mathfrak{I}}_2}(H) \neq \tau_{R_q}^{\hat{\mathfrak{I}}_3}(H) \,\forall\; q \in \{1, 2, 3\}.$$

Also, in this case

$$\tau_{R_q}^{<3\hat{\mathfrak{I}}>-\hat{\mathfrak{I}}_1}(H) = \tau_{R_q}^{<\hat{\mathfrak{I}}_2,\hat{\mathfrak{I}}_3>}(H) \neq \tau_{R_q}^{<3\hat{\mathfrak{I}}>}(H) \,\forall\; q \in \{1, 2, 3\}.$$
$$\tau_{R_q}^{<3\hat{\mathfrak{I}}>-\hat{\mathfrak{I}}_2}(H) = \tau_{R_q}^{<\hat{\mathfrak{I}}_1,\hat{\mathfrak{I}}_3>}(H) \neq \tau_{R_q}^{<3\hat{\mathfrak{I}}>}(H) \,\forall\; q \in \{1, 2, 3\}.$$
$$\tau_{R_q}^{<3\hat{\mathfrak{I}}>-\hat{\mathfrak{I}}_3}(H) = \tau_{R_q}^{<\hat{\mathfrak{I}}_1,\hat{\mathfrak{I}}_2>}(H) = \tau_{R_q}^{<3\hat{\mathfrak{I}}>}(H) \,\forall\; q \in \{1, 2, 3\}.$$

$\Longrightarrow$ $\hat{\mathfrak{I}}_3$ is insignificant when relation $R = R_1, R_2, R_3$ when the conditional attributes are taken under consideration one by one. Also, in this case, single-ideals $\hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2$ are neither insignificant nor reliable when compared to the ideal generated by multiple ones. However, when merged to generate a multi-ideal, all three ideals yield more accurate and reliable results. (Calculation is skipped as it is quite cumbersome.) Hence, we can conclude that relying on a single ideal (representing an expert opinion) is not sufficient or efficient. It's evident from this example that no individual ideal is adequate for thoroughly analyzing the decision; this is where multiple ideals prove their significance. This underscores the importance and utility of using multiple ideals rather than a single one in feature selection and multi-criteria decision-making. By incorporating more than one ideal in rough sets, the granularity of analysis improves, allowing a more structured interpretation of information. This multi-ideal approach enables the identification and elimination of inconsistencies, leading to more precision within rough sets. Enhanced accuracy is well-illustrated in Table 2. Clearly,

$$\mu_R(H) \leq \mu^{\hat{\mathfrak{I}}_i}{}_{R_j}(H) \leq \mu^{<3\hat{\mathfrak{I}}>}{}_{R_j}(H)$$

Table 2 Comparison of accuracies of different rough approximations of multi-ideal to single-ideal approach

Relation (R)	$\mu_R(H)$	$\mu^{\hat{\Im}_1}{}_R(H)$	$\mu^{\hat{\Im}_2}{}_R(H)$	$\mu^{\hat{\Im}_3}{}_R(H)$	$\mu^{<3\hat{\Im}>}{}_R(H)$
R_1	0	$\frac{2}{5}$	0	0	$\frac{2}{3}$
R_2	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{1}{4}$	1
R_3	0	1	$\frac{3}{4}$	0	1
R_4	1	1	1	1	1

Table 3 A comparison of the upper approximations of Z of the introduced ideology with the previous ones

Z	$U_R(Z)$	$\overline{R_j}^{\hat{\Im}_1}(Z)$	$\overline{R_j}^{\hat{\Im}_2}(Z)$	$\overline{R_j}^{<n\hat{\Im}>}(Z)$
$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$
$\{\upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$
$\{\upsilon_1, \upsilon_2\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1\}$
$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$\{\upsilon_1, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	$\{\upsilon_1, \upsilon_4\}$	$\{\upsilon_1, \upsilon_4, \upsilon_3\}$	$\{\upsilon_1, \upsilon_4\}$
$\{\upsilon_2, \upsilon_3\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3\}$
$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$
$\{\upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$
$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1, \upsilon_3\}$
$\{\upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$
$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_4\}$
$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$
$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$

$\forall$ i=1,2,...,3 and $\forall$ $R = R_1, R_2, R_3, R_4$ as shown in Table 2.

We construct the next two examples to clarify the preference of the proposed approach compared to the foregoing ones in terms of upper approximation, lower approximation, boundary region, and accuracy measure. The example 4.2 is built with respect to an equivalence relation and the next one 4.3 is structured with respect to a non-equivalence relation.

Example 4.2 Let $U = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$. Also, let $R = \{\{\upsilon_1, \upsilon_1\}, \{\upsilon_2, \upsilon_2\}, \{\upsilon_3, \upsilon_3\}, \{\upsilon_4, \upsilon_4\}\}$, $\{\upsilon_1, \upsilon_3\}, \{\upsilon_3, \upsilon_1\}$; it is clear that R is an equivalence relation. Let us consider that $\hat{\Im}_1 = \{\emptyset, \{\upsilon_1\}\}$ and $\hat{\Im}_2 = \{\emptyset, \{\upsilon_4\}\}$. Now, $< 2\hat{\Im} > = < \hat{\Im}_1, \hat{\Im}_2 > = \{\emptyset, \{\upsilon_1\}, \{\upsilon_4\}, \{\upsilon_1, \upsilon_4\}\}$. As relation is equivalence, $R_l(k) = R_r(k) = R_i(k) = R_u(k)$ for every $k \in \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$. Comparison of our introduced approach is done with the previous ones in Tables 3, 4, 5 and 6:

Example 4.3 Let $U = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$. Also, let $R = \{\{\upsilon_1, \upsilon_1\}, \{\upsilon_1, \upsilon_2\}, \{\upsilon_2, \upsilon_4\}, \{\upsilon_3, \upsilon_1\}\}, \{\upsilon_3, \upsilon_4\}, \{\upsilon_4, \upsilon_3\}, \{\upsilon_4, \upsilon_2\}, \{\upsilon_2, \upsilon_2\}, \{\upsilon_3, \upsilon_3\}, \{\upsilon_4, \upsilon_4\}$. Note that R is reflexive but non-equivalence relation. $\hat{\Im}_1 = \{\emptyset\}$. $\hat{\Im}_2 = \{\emptyset, \{\upsilon_1\}, \{\upsilon_4\}, \{\upsilon_1, \upsilon_4\}\}$. $\hat{\Im}_3 = \{\emptyset, \{\upsilon_4\}\}$.

Table 4 A comparison of the lower approximations of Z of the introduced ideology with the previous ones

Z	$L_R(Z)$	$\underline{R_j}^{\hat{\Im}_1}(Z)$	$\underline{R_j}^{\hat{\Im}_2}(Z)$	$\underline{R_j}^{nI}(Z)$
$\{\upsilon_1\}$	$\{\upsilon_1\}$	$\{\emptyset\}$	$\emptyset$	$\emptyset$
$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$
$\{\upsilon_3\}$	$\{\upsilon_3\}$	$\{\upsilon_3\}$	$\emptyset$	$\{\upsilon_3\}$
$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$
$\{\upsilon_1, \upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$
$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$\{\upsilon_1, \upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$
$\{\upsilon_2, \upsilon_3\}$	$\{\upsilon_2, \upsilon_3\}$	$\{\upsilon_2\}$	$\{\upsilon_2\}$	$\{\upsilon_2, \upsilon_3\}$
$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$
$\{\upsilon_3, \upsilon_4\}$	$\{\upsilon_3, \upsilon_4\}$	$\{\upsilon_3, \upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_3, \upsilon_4\}$
$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1, \upsilon_3\}$	$\{\upsilon_2, \upsilon_1, \upsilon_3\}$
$\{\upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_3\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_3, \upsilon_4\}$
$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$	$\{\upsilon_2, \upsilon_4\}$
$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$
$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$

Table 5 A comparison of the boundary of Z of the introduced ideology with the previous ones

Z	$B_R(Z)$	$B^{\hat{\Im}_1}{}_{R_j}(Z)$	$B^{\hat{\Im}_2}{}_{R_j}(Z)$	$B^{<n\hat{\Im}>}{}_{R_j}(Z)$
$\{\upsilon_1\}$	$\{\upsilon_3\}$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\upsilon_3\}$	$\{\upsilon_1\}$	$\{\upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\upsilon_1, \upsilon_2\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_1, \upsilon_3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\upsilon_1, \upsilon_4\}$	$\{\upsilon_1, \upsilon_2\}$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_2, \upsilon_3\}$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_2, \upsilon_4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\upsilon_3, \upsilon_4\}$	$\{\upsilon_1\}$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\upsilon_2, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	$\emptyset$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$< 3\hat{\Im} > \; = \; < \hat{\Im}_1, \hat{\Im}_2, \hat{\Im}_3 >= \{\emptyset, \{\upsilon_1\}, \{\upsilon_4\}, \{\upsilon_1, \upsilon_4\}\}$. Note that if the relation is non-equivalence, the standard definition of nano-topology is not applicable. Also, in general, $R_l(k) \neq R_r(k) \neq R_i(k) \neq R_u(k)$ for every $k \in \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$.

A comparison of different neighborhoods is done with the previous ones in Tables 7 and 8 with respect to a subset $Z = \{\upsilon_1, \upsilon_3\}$.

Table 6 A comparison of the accuracy of Z of the introduced ideology with the previous ones

Z	$\mu_R(Z)$	$\mu^{\hat{\Im}_1}{}_{R_j}(Z)$	$\mu^{\hat{\Im}_2}{}_{R_j}(Z)$	$\mu^{<n\hat{\Im}>}{}_{R_j}(Z)$
$\{\upsilon_1\}$	$\frac{1}{2}$	0	0	0
$\{\upsilon_2\}$	1	1	1	1
$\{\upsilon_3\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\{\upsilon_4\}$	1	1	1	1
$\{\upsilon_1, \upsilon_2\}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
$\{\upsilon_1, \upsilon_3\}$	1	1	1	1
$\{\upsilon_2, \upsilon_3\}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
$\{\upsilon_1, \upsilon_4\}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
$\{\upsilon_2, \upsilon_4\}$	1	1	1	1
$\{\upsilon_3, \upsilon_4\}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
$\{\upsilon_1, \upsilon_2, \upsilon_3\}$	1	1	1	1
$\{\upsilon_2, \upsilon_3, \upsilon_4\}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$\{\upsilon_1, \upsilon_2, \upsilon_4\}$	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\frac{3}{4}$	1	1	1
$\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$	1	1	1	1

Table 7 A comparison of the upper, lower approximations and boundaries of Z of the introduced topology with the previous notions for j-neighborhoods, where $j \in \{r, i, l, u\}$

j	r	l	i	u
$\overline{R_j}^{\hat{\Im}_1}(Z)$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$\overline{R_j}^{\hat{\Im}_2}(Z)$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3, \upsilon_4\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$\overline{R_j}^{\hat{\Im}_3}(Z)$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$\overline{R_j}^{<3\hat{\Im}>}(Z)$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$\underline{R_j}^{\hat{\Im}_1}(Z)$	$\{\upsilon_1\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\underline{R_j}^{\hat{\Im}_2}(Z)$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1\}$
$\underline{R_j}^{\hat{\Im}_3}(Z)$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$\underline{R_j}^{<3\hat{\Im}>}(Z)$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$	$\{\upsilon_1, \upsilon_3\}$
$B^{\hat{\Im}_1}{}_{R_j}(Z)$	$\{\upsilon_3\}$	$\emptyset$	$\emptyset$	$\{\upsilon_3\}$
$B^{\hat{\Im}_2}{}_{R_j}(Z)$	$\emptyset$	$\{\upsilon_4\}$	$\{\emptyset\}$	$\{\upsilon_3\}$
$B^{\hat{\Im}_3}{}_{R_j}(Z)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$B^{<n\hat{\Im}>}{}_{R_j}(Z)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 8 The comparison of the accuracy measures of Z of the suggested topology with the established ones for j-neighborhoods, where $j \in \{r, i, l, u\}$

j	r	l	i	u
$\mu^{<3\hat{\Im}>}{}_{R_j}(Z)$	1	1	1	1
$\mu^{\hat{\Im}_1}{}_{R_j}(Z)$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
$\mu^{\hat{\Im}_2}{}_{R_j}(Z)$	1	$\frac{2}{3}$	1	$\frac{1}{2}$
$\mu^{\hat{\Im}_3}{}_{R_j}(Z)$	1	1	1	1

In what follows, we present an algorithm for detecting the core from any information table from a given set of attributes.

Step I: If U is the universe set and H is a finite set of attributes that can further be categorized in two parts, H_1 of conditional attributes/factors and H_2 of decision-related attributes/factors, a binary relation R on U corresponding to H_1 and $X \subseteq U$, then tabularize the data, where columns are labelled as a particular set of attributes, rows are specified by entities and cells of the table are entries assigned to attributes according to the respective domains. Also, $\hat{\Im}_1, \hat{\Im}_2$ ....and $\hat{\Im}_n$ are the 'n' ideals on U, which signify the opinions of various known experts.

Step II : Using Definition 3.3, find the lower and upper approximation along with the boundary of X with respect to R, as per the notion of multi-ideal approximation space.

Step III : Induce the multi-ideal nano-topology using Definition 3.6 $\tau_{R_j}^{<n\hat{\Im}>}(X)$ on U.

Step IV : Remove a factor/attribute z from H_1. After removing that z, find the lower and upper approximations and hence the boundary of X with respect to the altered relation R' on $H_1 - \{z\}$.

Step V : Construct the multi-ideal nano-topology $\tau'_{R_j}{}^{<n\hat{\Im}>}(X)$ according to the modified data on U.

Step VI : Replicate steps IV and V for each attribute in H_1.

Step VII : The attributes in H_1 for which $\tau_{R_j}^{<n\hat{\Im}>}(X) \neq \tau'_{R_j}{}^{<n\hat{\Im}>}(X)$ comprise the collection of the most significant traits or characteristics, which is referred mathematically as core(R).

5 An application of multi-ideal nano-topology to diagnosis and treatment of dengue disease

The core, as mentioned in the algorithm represents the set of attributes essential for preserving the classification ability of a dataset. It is the intersection of all reducts and contains only indispensable attributes. Identifying the core helps simplify data analysis while maintaining critical information for decision-making. This section deals with a real-life example of dengue, where nano-topology is applied to determine the key symptoms responsible for the disease. Secondly, the most suitable medicine is suggested based on the given data. Throughout this application, as we are considering equivalence relation, so R_j neighborhoods are indiscernible, hence respective topologies are indistinguishable, which means $j \in \{l, r, i, u\}$.

Table 9 Illustration of patients with respect to various symptoms and reports

Patients	Rashes	Fever	Headache	Vomiting	Fatigue	Decision
ϱ_1	+	+	–	–	+	✓
ϱ_2	–	+	+	+	+	✓
ϱ_3	+	+	–	+	–	✓
ϱ_4	–	–	+	+	+	✗
ϱ_5	+	+	–	–	+	✗
ϱ_6	+	+	–	+	+	✗
ϱ_7	+	+	–	+	–	✓
ϱ_8	+	–	–	–	–	✗

5.1 Diagnosis of dengue disease

Table 9 demonstrates some patients $\{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7, \varrho_8\}$ with respect to different conditional attributes (symptoms) such as rashes, fever, headache, vomiting, and fatigue whereas as decision (dengue report) is the decision attribute. The domain of each attribute is given as {+, -} depending on whether the patient has that symptom or not. Also, the domain of the decision attribute is given as {✓, ✗}.

Now, the universe U is $\{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7, \varrho_8\}$. Let R be an equivalence relation on U with respect to the collection of all attributes, taken together. so $R = \{\{\varrho_1, \varrho_5\}, \{\varrho_3, \varrho_7\}, \{\varrho_2\}, \{\varrho_4\}, \{\varrho_6\}, \{\varrho_8\}\}$. Also, $\hat{\mathfrak{I}}_1 = \{\emptyset, \{\varrho_2\}, \{\varrho_3, \varrho_2\}, \{\varrho_3\}\}$ refers to the report of external expert 1, $\hat{\mathfrak{I}}_2 = \{\emptyset, \{\varrho_3\}, \{\varrho_7, \varrho_3\}, \{\varrho_7\}\}$ refers to the report of any external expert 2 and $\hat{\mathfrak{I}}_3 = \{\emptyset, \{\varrho_7\}\}$ refers to the report of any external expert 3. $<\hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2, \hat{\mathfrak{I}}_3> = \{\emptyset, \{\varrho_2\}, \{\varrho_3\}, \{\varrho_7\}\}, \{\varrho_2, \varrho_3\}, \{\varrho_3, \varrho_7\}, \{\varrho_2, \varrho_7\}, \{\varrho_2, \varrho_3, \varrho_7\}\}$ refers to the combined report of all three external health experts.

Let $X = \{\varrho_1, \varrho_2, \varrho_3, \varrho_7\}$ be the patients having positive report of dengue. So, $X^c = U - X = \{\varrho_4, \varrho_5, \varrho_6, \varrho_8\}$. By the definition, the nano-topology on U with respect to X is $\tau_{R_j}^{<\hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2, \hat{\mathfrak{I}}_3>}(X) = \tau_{R_j}^{<3\hat{\mathfrak{I}}>}(X) = \{U, \emptyset, \{\varrho_2, \varrho_3, \varrho_7\}, \{\varrho_2, \varrho_3, \varrho_7, \varrho_1, \varrho_5\}, \{\varrho_1, \varrho_5\}\}$.

Case 1: If the symptom "rashes" is neglected from the collection of conditional attributes, then $R - (rashes) = \{\{\varrho_1, \varrho_5\}, \{\varrho_3, \varrho_7\}, \{\varrho_2\}, \{\varrho_4\}, \{\varrho_6\}, \{\varrho_8\}\}$ Hence, topology induced by $R' = R - (rashes)$ is given by $\tau'_{R_j}{}^{<3\hat{\mathfrak{I}}>}(X) = \{U, \emptyset, \{\varrho_2, \varrho_3, \varrho_7\}, \{\varrho_2, \varrho_3, \varrho_7, \varrho_1, \varrho_5\}, \{\varrho_1, \varrho_5\}\} = \tau_{R_j}^{<3\hat{\mathfrak{I}}>}(X)$.

Case 2: If "fever" is removed from the collection of condition attributes, then $R - (fever) = \{\{\varrho_1, \varrho_5\}, \{\varrho_3, \varrho_7\}, \{\varrho_2, \varrho_4\}, \{\varrho_6\}, \{\varrho_8\}\}$. Hence, topology induced by $R' = R - (fever)$ is given by $\tau'_{R_j}{}^{<3\hat{\mathfrak{I}}>}(X) = \{U, \emptyset, \{\varrho_3, \varrho_7\}, \{\varrho_1, \varrho_2, \varrho_3, \varrho_5, \varrho_7\}, \{\varrho_1, \varrho_2, \varrho_5\}\} \neq \tau_{R_j}^{<3\hat{\mathfrak{I}}>}(X)$.

Case 3: If "headache" is excluded from the collection of attributes, then $R - (headache) = R$ and hence, topology induced by $R' = R - (headache)$ is given by $\tau'_{R_j}{}^{<n\hat{\mathfrak{I}}>}(X) = \tau_{R_j}^{<n\hat{\mathfrak{I}}>}(X)$.

Case 4: If "vomiting" is neglected from the collection of attributes, then $R' = R - (vomitting) = \{\{\varrho_1, \varrho_5, \varrho_6\}, \{\varrho_2\}, \{\varrho_3, \varrho_7\}, \{\varrho_4\}, \{\varrho_8\}\}$ and the topology induced by R' is given by $\tau'_{R_j}{}^{<3\hat{\mathfrak{I}}>}(X) = \tau_{R_j}{}^{<3\hat{\mathfrak{I}}>}(X)$.

Table 10 Illustration of medicines with respect to various conditions and reports

Medicines	cost	Dosage	A.R.T	R.O.A	side-effects	chemical/herbal	(Recomm.)
ϑ_1	low	medium	7–15	Oral	Negligible	Herbal	✓
ϑ_2	high	heavy	≥ 15	Parenteral	Negligible	Chemical	✗
ϑ_3	low	light	≤ 7	Oral	Negligible	Herbal	✗
ϑ_4	high	light	7–15	Oral	Significant	Ayurvedic	✓
ϑ_5	high	heavy	≥ 15	Parenteral	Significant	Chemical	✗
ϑ_6	low	light	≤ 7	Oral	Negligible	Herbal	✗
ϑ_7	low	light	7-15	Oral	Negligible	Herbal	✓
ϑ_8	high	heavy	≤ 7	Parenteral	Significant	Chemical	✗
ϑ_9	high	heavy	≥ 15	Parenteral	Significant	Chemical	✓
ϑ_{10}	low	light	≤ 7	Oral	Negligible	Herbal	✓

Case 5: If "fatigue" is neglected from the factors, then $R-(fatigue) = \{\{\varrho_1, \varrho_5\}, \{\varrho_2\}, \{\varrho_3, \varrho_6, \varrho_7\}, \{\varrho_4\}, \{\varrho_8\}\}$ and hence topology induced by $R' = R - (fatigue)$ is given by $\tau'^{<3\hat{\Im}>}_{R_j}(X) = \{U, \emptyset, \{\varrho_3, \varrho_7\}, \{\varrho_1, \varrho_2, \varrho_3, \varrho_5, \varrho_7\}, \{\varrho_1, \varrho_2, \varrho_5\}\} \neq \tau_{R_j}^{<3\hat{\Im}>}(X)$.

According to the above computations, we get that the symptoms of fever, and fatigue are the core of R; i.e., Core (R)={Fever, Fatigue}. Hence, we arrive at the conclusion that "fever" and "fatigue" are the key symptoms to judge whether the patient has dengue disease or not.

5.2 Treatment of dengue disease

Secondly, we find the best suitable or the most appropriate medicine which can cure the disease of dengue depending upon the various conditional attributes. Consider Table 10.

Take $X = \{\vartheta_1, \vartheta_4, \vartheta_7, \vartheta_9, \vartheta_{10}\}$ be the five medicines which are recommended by a health specialist and $X^c = \{\vartheta_2, \vartheta_3, \vartheta_5, \vartheta_6, \vartheta_8\}$ be the set of medicines which have not been recommended by a health specialist. Here, $\hat{\Im}_1 = \{\emptyset, \{\vartheta_1\}, \{\vartheta_3\}, \{\vartheta_1, \vartheta_3\}\}$, $\hat{\Im}_2 = \{\emptyset, \{\vartheta_5\}, \{\vartheta_3\}, \{\vartheta_9\}, \{\vartheta_3, \vartheta_9\}, \{\vartheta_5, \vartheta_9\}, \{\vartheta_3, \vartheta_9\}, \{\vartheta_3, \vartheta_5, \vartheta_9\}\}$ and $\hat{\Im}_3 = \{\emptyset, \{\vartheta_9\}\}$. $< \hat{\Im}_1, \hat{\Im}_2, \hat{\Im}_3 > = \{\emptyset, \{\vartheta_1\}, \{\vartheta_3\}, \{\vartheta_5\}, \{\vartheta_9\}, \{\vartheta_3, \vartheta_5\}, \{\vartheta_5, \vartheta_9\}, \{\vartheta_3, \vartheta_9\}, \{\vartheta_1, \vartheta_3\}, \{\vartheta_1, \vartheta_9\}, \{\vartheta_1, \vartheta_5\}, \{\vartheta_1, \vartheta_3, \vartheta_5\}, \{\vartheta_1, \vartheta_3, \vartheta_9\}, \{\vartheta_1, \vartheta_5, \vartheta_9\}, \{\vartheta_3, \vartheta_5, \vartheta_9\}, \{\vartheta_1, \vartheta_3, \vartheta_5, \vartheta_9\}\}$. Here, $\hat{\Im}_1$, $\hat{\Im}_2$ and $\hat{\Im}_3$ represent the prescription of medicines by external doctors 1, 2, and 3 respectively. $< \hat{\Im}_1, \hat{\Im}_2, \hat{\Im}_3 >$ i.e. ideal generated by $\hat{\Im}_1$, $\hat{\Im}_2$ and $\hat{\Im}_3$ means the collaborative report of three expert doctors. Note that the linguistic terms (such as high, low, heavy, etc) in an information table represent qualitative data, facilitating the understanding of attributes in natural language. They enable more intuitive data analysis, improve the interpretability of results, and bridge the gap between human reasoning and computational processes, thus enhancing data interpretation in complex systems. Usually, we don't have specified quantitative data, so it is more useful to use these terms to study the interrelation of various attributes and entities.

Here,

(i) The decision attributes are $\{\checkmark, \times\}$ depending upon whether recommended or not.
(ii) The domain of attribute (cost) = {low, high}.
(iii) Domain of attribute (dosage) = {heavy, medium, light}.
(iv) Domain of attribute(average recovery time or A.R.T) = $\{7 - 15, \geq 15, \leq 7\}$ (in days).

(v) Domain of attribute (route of administration) = {oral, parenteral}.
(vi) Domain of attribute(side-effects) = {negligible, significant}.
(vii) The domain of attribute (nature of medicine) = {herbal, chemical, ayurvedic}.

Now, if R is the indiscernibility relation, it means we say that two medicines are related if they are indistinguishable from a given set of attributes. Here, $R = \big\{\{\vartheta_1\}, \{\vartheta_2\}, \{\vartheta_3, \vartheta_6, \vartheta_{10}\}, \{\vartheta_4\}, \{\vartheta_5, \vartheta_9\}, \{\vartheta_7\}, \{\vartheta_8\}\big\}$.Throughout, $j \in \{l, r, i, u\}$. If $\hat{\Im}_1, \hat{\Im}_2, \hat{\Im}_3$ given and $X \subseteq U$, then multi-ideal nano-topology can be given by $\tau_{R_j}^{<3\hat{\Im}>}(X) = \{\emptyset, U, \underline{R_j}^{<3\hat{\Im}>}(X), \overline{R_j}^{<3\hat{\Im}>}(X), B^{<3\hat{\Im}>}{}_{R_j}(X)\} = \{\emptyset, U, \{\vartheta_1, \vartheta_4, \vartheta_7, \vartheta_9\}, \{\vartheta_1, \vartheta_4, \vartheta_3, \vartheta_6, \vartheta_7, \vartheta_9, \vartheta_{10}\}, \{\vartheta_3, \vartheta_6, \vartheta_{10}\}$.

Case 1: Firstly, when we neglect an attribute "cost" from the set of attributes, then we have $R' = R-\{\text{cost}\}$. So, $R' = \big\{\{\vartheta_1\}, \{\vartheta_2\}, \{\vartheta_3, \vartheta_6, \vartheta_{10}\}, \{\vartheta_4\}, \{\vartheta_5, \vartheta_9\}, \{\vartheta_7\}, \{\vartheta_8\}\big\} = R$. Hence, $\tau_{R_j}^{'<3\hat{\Im}>}(X)$ is induced by $R' = R-\{\text{cost}\}$ is equal to $\tau_{R_j}^{<3\hat{\Im}>}(X)$, i.e., $\tau_{R_j}^{'<3\hat{\Im}>}(X) = \{\emptyset, U, \underline{R'_j}^{<3\hat{\Im}>}(X), \underline{R'_j}^{<3\hat{\Im}>}(X), B^{<3\hat{\Im}>}{}_{R'_j}(X)\} = \tau_{R_j}^{<3\hat{\Im}>}(X)$ as $R' = R$.

Case 2: Secondly, neglecting "dosage" from the attributes' set, then we have $R' = R-\{\text{dosage}\}$. So, $R' = R-\{\text{dosage}\}$ which implies that $R' = \big\{\{\vartheta_1, \vartheta_7\}, \{\vartheta_2\}, \{\vartheta_3, \vartheta_6, \vartheta_{10}\}, \{\vartheta_4\}, \{\vartheta_5, \vartheta_9\}, \{\vartheta_8\}\big\} \neq R$. But, $\tau_{R_j}^{'<3\hat{\Im}>}(X)$ (with respect to R') is given by $\tau_{R_j}^{'<3\hat{\Im}>}(X) = \{\emptyset, U, \{\vartheta_1, \vartheta_4, \vartheta_7, \vartheta_9\}, \{\vartheta_1, \vartheta_4, \vartheta_3, \vartheta_6, \vartheta_7, \vartheta_9, \vartheta_{10}\}, \{\vartheta_3, \vartheta_6, \vartheta_{10}\} = \tau_{R_j}^{<3\hat{\Im}>}(X)$.

Case 3: Then, neglecting "average recovery time" (A.R.T) from the attributes' set, then we have $R' = R-\{A.R.T\}$. So, $R' = \big\{\{\vartheta_1\}, \{\vartheta_2\}, \{\vartheta_3, \vartheta_6, \vartheta_7, \vartheta_{10}\}, \{\vartheta_4\}, \{\vartheta_5, \vartheta_8, \vartheta_9\}\big\} \neq R$. Hence, $\tau_{R_j}^{'<3\hat{\Im}>}(X) = \{\emptyset, U, \{\vartheta_1\}, \{\vartheta_1, \vartheta_3, \vartheta_4, \vartheta_6, \vartheta_7, \vartheta_9, \vartheta_{10}\}, \{\vartheta_3, \vartheta_4, \vartheta_6, \vartheta_7, \vartheta_9, \vartheta_{10}\}\} \neq \tau_{R_j}^{<3\hat{\Im}>}(X)$.

Case 4: On removing factor "Route of administration" (R.O.A) from the attributes' set, then we have $R' = R-\{R.O.A\} = \big\{\{\vartheta_1\}, \{\vartheta_2\}, \{\vartheta_3, \vartheta_6, \vartheta_{10}\}\}, \{\vartheta_4\}, \{\vartheta_5, \vartheta_9\}, \{\vartheta_7\}, \{\vartheta_8\}\big\} = R$. Hence, $\tau_{R_j}^{'<3\hat{\Im}>}(X)$ is induced by $R' = R - \{R.O.A\}$ is equal to $\tau_{R_j}^{<3\hat{\Im}>}(X)$, i.e., $\tau_{R_j}^{'<3\hat{\Im}>}(X) = \tau_{R_j}^{<3\hat{\Im}>}(X)$ as $R' = R$.

Case 5: Neglecting "side-effects" (S.E.) from the attributes' set, then we have $R' = R-\{\text{S.E}\}$. So, $R' = \big\{\{\vartheta_1\}, \{\vartheta_2, \vartheta_5, \vartheta_9\}, \{\vartheta_3, \vartheta_6, \vartheta_{10}\}, \{\vartheta_4\}, \{\vartheta_7\}, \{\vartheta_8\}\big\} \neq R$. Hence, $\tau_{R_j}^{'<3\hat{\Im}>}(X) = \{\emptyset, U, \{\vartheta_1, \vartheta_4, \vartheta_7\}, \{\vartheta_1, \vartheta_3, \vartheta_4, \vartheta_6, \vartheta_7, \vartheta_9, \vartheta_{10}\}, \{\vartheta_3, \vartheta_6, \vartheta_9, \vartheta_{10}\}\} \neq \tau_{R_j}^{<3\hat{\Im}>}(X)$.

Case 6: Lastly, neglecting "Chemical/herbal/ayurvedic" (C./H./A) from the attributes' set, then we have $R' = R-\{(\text{C./H./A})\}$. So, $R' = \big\{\{\vartheta_1\}, \{\vartheta_2\}, \{\vartheta_3, \vartheta_6, \vartheta_{10}\}, \{\vartheta_4\}, \{\vartheta_5, \vartheta_9\}, \{\vartheta_7\}, \{\vartheta_8\}\big\} = R$. Hence, $\tau_{R_j}^{'<3\hat{\Im}>}(X)$ is induced by $R' = R-\{(\text{C./H./A})\}$ is equal to $\tau_{R_j}^{<3\hat{\Im}>}(X)$, i.e., $\tau_{R_j}^{'<3\hat{\Im}>}(X) = \tau_{R_j}^{<3\hat{\Im}>}(X)$ as $R' = R$.

So, we conclude that the collection of attributes for which $\tau_{R_j}^{'<3\hat{\Im}>}(X) \neq \tau_{R_j}^{<3\hat{\Im}>}(X)$ is {average recovery time, side-effects}. Hence, the above-listed attributes are the most imperative in deciding the most appropriate medicines to test on many similar patients. That is, core(R)= {average recovery time, side-effects}. Hence, medicines with light short average recovery time, and negligible side-effects are the best for dengue patients.

In this hypothetical information system, based on the above observation, the most suitable medicines for the cure of dengue disease = $\{\vartheta_3, \vartheta_6, \vartheta_{10}\}$.

6 Conclusion and future scope

Introduced in 2013 by Thivagar and Richard (2013), nano-topology holds a huge potential in real-life situations. It can be applied effectively to medicine, information sciences, research, and technology. In the last ten years, different notions have been studied to generalize the original definition of nano-topology. Nano-topology has thoroughly been investigated via different approximations using different mathematical tools such as ideals and neighborhoods (Kandil et al. 2021).

In this article, we have proposed a novel technique to compute the approximation operators of subsets using a finite set of ideals instead of one ideal aiming to heighten the accuracy measures of subsets. We have explored the fundamentals of present rough models and furnished some examples to clarify their merits compared to some preceding rough set models. Then, we applied this technique to establish the concept of multi-ideal nano-topology and discussed its main properties. We have illustrated that the structures of nano-topology and ideal nano-topology are special cases of multi-ideal nano-topology when ideals are empty or equal. In the end, we have examined the frame of multi-ideal nano-topology to diagnose and treat dengue disease and by the given discussion and analysis we showed that multi-ideal nano-topology has helped in devising an algorithm to study the inter-dependency of various factors/conditional attributes and decision attributes if provided the opinions of multiple experts or reports.

The proposed technique can be used to analyze and simplify complex intelligent systems. It can be used to study the quantitative as well as qualitative data. It has scope in various fields like medicine, biochemistry, and engineering. Also, in upcoming work, we will make use of the current approach to approximate subsets using other types of neighborhood systems and topological spaces (such as supra topology and infra topology).

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